

THE UNORIENTED COBORDISM CLASSES OF COMPACT FLAT RIEMANNIAN MANIFOLDS

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This paper provides partial information on the conjecture that all compact flat riemannian manifolds are boundaries. The main result is

Theorem 3.1. *If ϕ is a finite 2-group whose elements of order two lie in its center, then any compact flat riemannian manifold with holonomy group ϕ is a boundary.*

Our approach is the study of certain canonical Z_2^k -actions called translational involutions. These involutions arise as the projections of translations of R^n . Theorem 3.1 is proved by first showing that the group of translational involutions has no stationary point and then appealing to the following well-known theorem of Conner and Floyd.

Theorem (Conner & Floyd [3, p. 76, Theorem 30.1]). *If Z_2^k acts differentiably on the closed n -manifold M^n without stationary points, then M^n is an unoriented boundary.*

In §1 we prove Lemma 1.2 which relates the assumption that the group of translational involutions does have a stationary point to the two-rank of the holonomy group. Immediate corollaries provide, in §2 bounds for each finite 2-group ϕ on both the dimension and the first betti number of a nonbounding flat manifold with holonomy group ϕ .

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1. Translational involutions

Recall that an n -dimensional compact flat riemannian manifold is the quotient of R^n by the action of a torsion-free Bieberbach group (see Charlap [1]), i.e., by the action of a subgroup π of the group of rigid motions of R^n satisfying the following two defining conditions: (i) π contains no elements of finite order, and (ii) there is an exact sequence

$$(1) \quad 0 \rightarrow L \rightarrow \pi \xrightarrow{\rho} \phi \rightarrow 0$$

in which L is an n -dimensional lattice of pure translations of R^n and where ϕ is a finite group which is isomorphic to the holonomy group of the quotient R^n/π . We note that the action of ϕ on L induced by conjugation is faithful, a fact which will be crucial to our arguments. Henceforth L will be viewed as a ϕ -module via this action.

One knows (see Charlap and Vasques [2]) that for each torsion-free Bieberbach group [1], the set $(L/2L)^\phi$ of ϕ -invariant elements of $L/2L$ is isomorphic to a subgroup of the group of affinities of R^n/π . Geometrically, this action is realized as the set of projections from R^n to R^n/π of the translations

$$(2) \quad x \rightarrow x + 1/2v$$

where v ranges over all lifts to L of elements of $(L/2L)^\phi$. Since $(L/2L)^\phi$ is elementary abelian, this action is a group of involutions.

Definition 1.1. The Z_2^k -action determined by $(L/2L)^\phi$ is called the group of translational involutions and is denoted by $I(\pi)$. For $W \in (L/2L)^\phi$, the corresponding involution is denoted by $I(W)$.

We adopt the convention that elements of π are represented by pairs (T, σ) where T is a translation and $\sigma \in O(n)$. For $x \in R^n$,

$$(T, \sigma) \cdot x = T + \sigma \cdot x.$$

The next lemma investigates the assumption that a subgroup of $I(\pi)$ has a stationary point.

Lemma 1.1. *Let G be a nontrivial subgroup of $(L/2L)^\phi$, and suppose that the subgroup of $I(\pi)$ which G determines has a stationary point x . Then there is an injective function $s: G \rightarrow \pi$, such that the composed map*

$$(3) \quad J = \rho \circ s : G \rightarrow \phi$$

is an injective homomorphism. Further, the function s may be chosen to be of the form,

$$(4) \quad s(W) = (1/2v, \sigma_W),$$

where v is some lift of W to L .

Proof. By change of coordinates in R^n we may assume that the origin covers the stationary point x . Choose a list $W_i, i = 1, p$, of the elements of G . Let $v_i \in L$ such that translation by $1/2v_i$ projects to the involution $I(W_i)$. x stationary implies that for each i there exists a unique element (T_i, σ_i) of π such that

$$0 + 1/2v_i = (T_i, \sigma_i) \cdot 0.$$

Note that $1/2v_i = T_i \forall i$. One defines the map s by $s(w_i) = (T_i, \sigma_i) \cdot s$ is

injective because $1/2v_i \neq 1/2v_j$. We next show that J is an injective homomorphism. Note that $J(W_i) = \rho((T_i, \sigma_i)) = \sigma_i$.

J is injective: $\sigma_i = \sigma_j, i \neq j$, implies that $T_i - T_j$ is an element of L because by (1), any two translations associated to an element of ϕ differ by a lattice point. Hence from the above, $1/2v_i - 1/2v_j$ is in L . But then, $I(W_i)$ equals $I(W_j)$. This is impossible because $W_i \neq W_j$. Thus J is injective.

J is a homomorphism: Since $1/2v_i = T_i$ for all i , we compute that

$$(T_i, \sigma_i) \cdot (T_j, \sigma_j) = (1/2v_i + 1/2v_j + l, \sigma_i\sigma_j)$$

for some $l \in L$ (since v_j is ϕ -invariant mod $2L$). Clearly, translation by $1/2(v_i + v_j)$ projects to $I(W_i + W_j)$. Since any two lifts of an element of $(L/2L)^\phi$ to L differ by an element of $2L$, if $1/2v_k$ was the above previously chosen translation representing $W_i + W_j$, then $1/2v_k - 1/2(v_i + v_j)$ is in L . Hence $1/2v_k$ is a translation associated to $\sigma_i\sigma_j$. Further, $\sigma_i\sigma_j$ is the only element of ϕ with translation $1/2v_k$, because π acts freely on R^n . Thus

$$J(W_i + W_j) = \sigma_i\sigma_j = J(W_i) \cdot J(W_j).$$

2. Two general theorems

In this section the following two theorems are proved.

Theorem 2.1. *Let R^n/π be a ϕ -manifold where ϕ is a finite 2-group. Suppose that the first betti number of R^n/π exceeds the 2-rank of ϕ . Then R^n is a boundary.*

Theorem 2.2. *Let R^n/π be a ϕ -manifold where ϕ is a finite 2-group of order 2^p and of 2-rank k . Then if n is greater than $k \cdot 2^p$, R^n/π is a boundary.*

Lemma 2.1. *If the Z_2 -dimension of $(L/2L)^\phi$ exceeds the 2-rank of ϕ , then R^n/π is a boundary.*

Proof. $\text{Dim}_{Z_2}((L/2L)^\phi)$ greater than the 2-rank of ϕ implies that the homomorphism J of Lemma 1.1 cannot be injective. Thus $I(\pi)$ has no stationary point, so the Lemma follows from Theorem 30.1 of Conner and Floyd [3].

Proof of Theorem 2.1. If the first betti number β_1 of R^n/π equals k , then L contains a k -dimensional trivial ϕ -submodule (see Wolf [7, p. 81]). Hence the Z_2 -dimension of $(L/2L)^\phi$ is no less than β_1 , and so is strictly greater than the 2-rank of ϕ . Now use Lemma 2.1.

Proof of Theorem 2.2. The theorem will follow from Lemma 2.1, once it has been shown that the Z_2 -dimension of $(L/2L)^\phi$ is greater than or equal to $n/2^p$. Let σ be a nontrivial element of order two in the center of ϕ . As a σ -module, $L/2L$ is the direct sum of trivial factors and Z_2 -group rings. Since

the action of Z_2 on its Z_2 -group ring has a unique fixed point, $\dim_{Z_2}(L/2L)^\sigma > n/2$. Now $(L/2L)^\sigma$ is a ϕ -module because σ is in the center of ϕ . Thus we may repeat this argument for $\phi/\langle\sigma\rangle$ and so on.

3. The main theorem

A finite 2-group all of whose elements of order two lie in its center will be called central. The main result is

Theorem 3.1. *If R^n/π is a ϕ -manifold where ϕ is a central 2-group, then R^n/π is a boundary.*

Corollary 3.1. *If R^n/π is a ϕ -manifold where ϕ is a finite group whose Sylow 2-subgroups are central, then R^n/π is a boundary.*

Proof. Let ψ be a Sylow 2-subgroup of ϕ , and let π' be its inverse image in π . The holonomy group of R^n/π' is ψ , so R^n/π' bounds by Theorem 3.1. Since the index of π' in π is odd, R^n/π' is an odd-fold covering space of R^n/π . But odd-fold covers preserve Stiefel-Whitney numbers, so R^n/π bounds as well (see Milnor & Stasheff [6]). q.e.d.

The proof of Theorem 3.1 consists in using purely algebraic facts (Propositions 3.1 and 3.2 below) concerning the cohomology of central 2-groups with coefficients in a lattice to deduce that a certain subgroup of the group of translational involutions has no stationary point. Specifically it will be shown that the assumption that there exists a stationary point implies that π contains an element of order two. The theorem then follows by contradiction and Theorem 30.1 of Conner and Floyd [3].

We turn next to the statements of Propositions 3.1 and 3.2, but postpone their proofs until these propositions have first been applied to proving Theorem 3.1.

A pure submodule A of a ϕ -module M will be said to be maximal if there exists no ϕ -module B such that $A \oplus B$ is isomorphic to a pure submodule of M .

Lemma 3.1. *Every nonzero finitely generated ϕ -module contains a nonzero maximal semisimple submodule.*

Proof. Obvious.

Definition 3.1. Let A be a k -dimensional semi-simple ϕ -module where ϕ is an elementary abelian 2-group of rank k . An extension

$$(5) \quad 0 \rightarrow A \rightarrow G \rightarrow \phi \rightarrow 0$$

will be said to be proper if it can be obtained from the following construction. Let J be an isomorphism from $A/2A$ onto ϕ . Let $s: A/2A \rightarrow A$ be a section

of $A/2A$ in A , i.e., s is an injective function such that $r \circ s = \text{id}_{A/2A}$ where $r: A \rightarrow A/2A$ is the projection homomorphism. Define g in $\text{Hom}_\phi(B_1(\phi); 1/2A)$, the group of one-dimensional ϕ -cochains with coefficients in $1/2A$, by

$$(6) \quad g(a) = 1/2s(J^{-1}(a)).$$

Since the image of δg , the coboundary of g , is actually contained in A , δg is an element of $Z^2(\phi; A)$, the group of inhomogeneous ϕ -cocycles with coefficients in A . Thus δg determines an extension of A by ϕ . (5) is proper if it may be obtained from some such g in this way.

Proposition 3.1. *Let L be a faithful ϕ -module where ϕ is an elementary abelian 2-group of rank k . Let $0 \rightarrow A \rightarrow G \rightarrow \phi \rightarrow 0$ be a proper extension, and let $i: A \rightarrow L$ be an injective homomorphism of ϕ -modules such that: (i) $i(A)$ is maximal, and (ii) $L/i(A)$ is semisimple. Let $0 \rightarrow L \rightarrow G' \rightarrow \phi \rightarrow 0$ be the extension determined by this change of coefficients. Then G' contains an element of order two.*

Proposition 3.2. *Let ϕ be a finite central 2-group, and let M be a finitely generated faithful ϕ -module. Let C be a nonzero maximal semisimple ϕ -submodule, and let $C \approx \bigoplus S_i, i = 1, k$, be a decomposition of C into ϕ -irreducible direct factors. Then for each elementary abelian subgroup G of ϕ of rank at least k there exist a faithful G -submodule L of M and a semisimple G -submodule A of L such that:*

- (a) A is generated (over Z) by precisely those elements of $C - 2C$ which are ϕ -invariant mod $2M$;
- (b) C splits as a G -direct sum, $A \oplus B$;
- (c) If L' and A' are the projections of L and A into M/B , then A' is maximal in L' and L'/A' is semisimple;
- (d) $(L \cap B) \subset 2B$.

Proof of Theorem 3.1. Let $0 \rightarrow M \rightarrow \pi \rightarrow \phi \rightarrow 0$ be a torsion free Bieberbach group with ϕ central. Choose a nonzero ϕ -module C in M as in Proposition 3.2. By condition (a) of Proposition 3.2 the corresponding module $A/2A$ represents a subgroup of $I(\pi)$. Assuming that this subgroup has a stationary point x we conclude from Lemma 1.1 that there is an elementary abelian subgroup ϕ' of ϕ which is isomorphic to $A/2A$ via the homomorphism J . Assuming that the origin of R^n covers the fixed point, J may be written as $\rho \circ s$ where s is defined by (4). Setting $g: \phi' \rightarrow 1/2A$ by (6), we obtain a proper extension,

$$0 \rightarrow A \rightarrow G \rightarrow \phi' \rightarrow 0.$$

Clearly the induced extension

$$0 \rightarrow A' \rightarrow G' \rightarrow \phi' \rightarrow 0$$

is proper. Hence by (c) of Proposition 3.2 and Proposition 3.1, the extension

$$0 \rightarrow L' \rightarrow G'' \rightarrow \phi' \rightarrow 0$$

contains an element of order two. Writing G'' as the cartesian product $L' \times \phi'$ with multiplication given by

$$(7) \quad (w, \sigma) \cdot (w', \tau) = (w + \sigma \cdot w' + \delta g(\sigma, \tau), \sigma\tau),$$

we see that the element of order two is of the form

$$(w, \sigma) \quad \text{where } \sigma \neq \text{id}.$$

We have from (7)

$$(0, \text{id}) = (w, \sigma)^2 = (w + \sigma \cdot w + \delta g(\sigma, \sigma), \text{id}).$$

Hence $\sigma \cdot w + w = -\delta g(\sigma, \sigma)$. By condition 4) of Proposition 3.2, in L we have

$$\sigma \cdot w + w = -\delta g(\sigma, \sigma) + 2b \quad \text{where } b \text{ is in } B.$$

Since σ acts trivially on both $\sigma \cdot w + w$ and $\delta g(\sigma, \sigma)$, it acts trivially on b as well. This gives

$$\sigma \cdot (w - b) + (w - b) = -\delta g(\sigma, \sigma).$$

Thus $(w - b, \sigma)$ is of order two. From this contradiction we see that $I(\pi)$ has no stationary point and hence that R^n/π is a boundary. This completes the proof.

Proof of Proposition 3.1. The aim here is to locate an element γ in ϕ and an element x in L such that $\gamma \cdot x + x = \delta g(\gamma, \gamma)$. For then, viewing G' as the cartesian product $L \times \phi$ with multiplication given by

$$(v, \sigma) \cdot (w, \tau) = (v + \sigma \cdot w + \delta g(\sigma, \tau), \sigma\tau),$$

one sees that $(-x, \gamma)$ is of order two.

Since $L/i(A)$ is semisimple for all x in L , γ in ϕ , the projection $r(\gamma \cdot x + x)$ of $\gamma \cdot x + x$ to $L/2L$ may be viewed as an element of $i(A)/2i(A)$. Hence there is a homomorphism

$$P : L \rightarrow \text{Hom}(\phi, \phi)$$

defined by

$$(8) \quad P(x)(\gamma) = J(r(\gamma \cdot x + x)).$$

Lemma 3.2. *In order to prove Proposition 3.1 it is sufficient to show that there exist x in L and γ in ϕ such that*

$$(9) \quad P(x)(\gamma) = J(r(\delta g(\gamma, \gamma))).$$

Proof. It follows from (9) that

$$\gamma \cdot x + x = \delta g(\gamma, \gamma) + 2w \quad \text{for some } w \text{ in } L,$$

so that

$$\gamma \cdot (x - w) + (x - w) = \delta g(\gamma, \gamma),$$

since γ acts trivially on $\delta g(\gamma, \gamma)$.

Lemma 3.3. *Let M be a faithful semisimple ϕ -module where ϕ is an elementary abelian 2-group of rank k . Then there exist vectors $x_i, i = 1, k$, in M and a minimal set of generators $\sigma_i, i = 1, k$, of ϕ such that*

$$(10) \quad \begin{aligned} \sigma_i \cdot x_j &= x_j, & i \neq j, \\ &= -x_j, & i = j. \end{aligned}$$

Proof. We omit the details. One proceeds by induction on k . q.e.d.

Since ϕ is elementary abelian, there are direct sum decompositions $L/i(A) \approx \bigoplus 1_i, i(A) \approx \bigoplus t_j$, where the 1_i 's and t_j 's are one-dimensional. Since L is faithful, $L/i(A) \oplus i(A)$ is faithful as well. Hence by reindexing if necessary, we may assume that $\bigoplus_{1 \leq i \leq s} 1_i \oplus i(A)$ is a faithful ϕ -module where s equals the rank of the subgroup ψ of ϕ of elements which act trivially on $i(A)$. Apply Lemma 3.3 to ψ to get vectors x_1, \dots, x_s in $\bigoplus 1_i$ and a minimal set of generators $\sigma_1, \dots, \sigma_s$ of ψ such that σ_i is "dual" to x_i in the sense of (10). Split ϕ as $\psi \times K$ where K is the kernel of the action of ϕ on $\bigoplus 1_i$. Next apply Lemma 3.3 to K to get vectors $e_j, j = 1, k - s$, in $i(A)$ and a minimal set of generators $\tau_j, j = 1, k - s$, of K such that τ_j is "dual" to e_j in the sense of (10). Note that $\sigma_i \cdot t_j = t_j$ and $\tau_j \cdot x_i = x_i, \forall i, j$. We intend to choose lifts u_i of the x_i 's to L such that the homomorphisms $P(u_i)$ have a particularly nice form.

Lemma 3.4. *There exists lifts u_i of the x_i 's to L such that $\sigma_j \cdot u_i = u_i$ for $i \neq j, -u_i + w_i$ for $i = j$ for some $w_i \in i(A) - 2i(A)$ and such that $\tau_j \cdot u_i \equiv u_i, \text{ mod } i(A)$ for all i, j .*

Proof. Let T_i denote the ϕ -module generated by A together with an arbitrary lift of x_i to L . $L/i(A)$ semisimple implies

$$T_i \otimes Q \approx L_i \oplus (i(A) \otimes Q),$$

where 1_i is one-dimensional. Let v_i be a Z -generator of $T_i \cap L_i$. Clearly, $\sigma_i \cdot v_i = -v_i$. Since $i(A)$ is maximal, $(L_i \cap T_i) \oplus i(A)$ is not pure. Thus there exist $u_i \in T_i, a_i \in i(A)$, and $m \in Z - \{0, \pm 1\}$ such that $m \cdot u_i = v_i + a_i$ and $u_i \notin (L_i \cap T_i) \oplus i(A)$. Now

$$\sigma_i \cdot u_i = -1/mv_i + 1/m a_i = -u_i + 2/m a_i.$$

Note that $2/m a_i \in A$. In fact, $2/m a_i \in A - 2A$ for otherwise, $1/m a_i \in A$ which implies that $1/m v_i$ is in T_i , but this contradicts the fact that v_i is a Z -generator of $(L_i \cap T_i)$. Setting $w_i = 2/m a_i$ takes care of the case where

$i = j$. For $i \neq j$, there is a similar computation: $\tau_j \cdot u_i \equiv u_i, \text{ mod } i(A)$, by construction. q.e.d.

Letting ϕ^* denote the dual space of ϕ (considered as a Z_2 -vector space) and identifying $\phi^* \otimes \phi$ with $\text{Hom}(\phi, \phi)$ in the usual way, define h_i, g_j in $\text{Hom}(\phi, \phi)$ by

$$(11) \quad h_i = \sigma_i^* \otimes P(u_i)(\sigma_i), \quad g_j = \tau_j^* \otimes J(z_j),$$

where z_j is the projection to $i(A)/2i(A)$ of e_j .

Let F be the additive subgroup of $\text{Hom}(\phi, \phi)$ generated by the h_i 's and the g_j 's. Note that F is generated by homomorphisms which are dual to the generators σ_i and τ_j of ϕ . The importance of Lemma 3.4 is that none of the h_i 's are the zero homomorphism.

Lemma 3.5. *There exist f in F and γ in $\phi - 0$ such that $f(\gamma) = \gamma$.*

Proof. The lemma is a special case of the more general fact that if x_1, \dots, x_k is a basis of a Z_2 -vector space V , and if f_1, \dots, f_k are elements of $\text{Hom}(V, V)$ such that $f_i(x_i) \neq 0$ for all i and $f_i(x_j) = 0$ for all $i \neq j$, then for some linear combination of the f 's fixes a nonzero element a in V . We omit the details. q.e.d.

Using Lemma 3.5, let $f \in F, \gamma \in \phi - 0$ such that $f(\gamma) = \gamma$. Factor f as

$$(12) \quad f = \sum h_i + \sum g_j,$$

and remove any of the h 's and g 's which send γ to zero. Assuming that this has already been done we see that γ contains the factor

$$\sum \sigma_i + \sum \tau_j$$

which we write simply as $\sigma\tau$. From Lemma 3.4 and equations (8) and (11) it follows that

$$f(\gamma) = P\left(\sum u_i\right)(\sigma) + J\left(r\left(\sum e_j\right)\right),$$

where the indices i and j correspond to the indices in (12). This last equation is written more simply as

$$(13) \quad f(\gamma) = P(x)(\sigma) + J(r(e)).$$

By Lemma 3.2, the proof is completed with

Lemma 3.6. $P(x)(\gamma) = J(r(\delta g)(\gamma, \gamma))$.

Proof. From (6), (8), and (13), the equation $f(\gamma) = \gamma$, and the equation $s \cdot r = \text{id}_A, \text{ mod } 2A$, it follows that

$$(14) \quad 2g(\gamma) = \sigma \cdot x + x + e + 2w, \quad \text{where } w \in i(A).$$

Since $\gamma \cdot e = -e$ we get

$$(15) \quad \delta g(\gamma, \gamma) = 1/2(\tau \cdot (\sigma \cdot x + x) + \sigma \cdot x + x + 2(\gamma \cdot w + w)).$$

Since A is a direct sum of one-dimensional γ -modules, A splits as a direct sum $A^+ \oplus A^-$, where γ acts trivially on A^+ and by negation on A^- . Hence from (15) it follows that

$$(16) \quad \delta g(\gamma, \gamma) = (\sigma \cdot x + x)^+, \text{ mod } 2i(A).$$

The lemma now follows from (16) and the easily checked fact that $\gamma \cdot x + x = (\sigma \cdot x + x)^+$. q.e.d.

Proof of Proposition 3.2. The following fact will be needed.

Lemma 3.7. *Let ϕ be a central group acting on the Q -vector space V . Suppose that V is ϕ -irreducible. If $\sigma \in \phi$ is of order 2, then σ either acts trivially on v or $\sigma \cdot v = -v$ for all $v \in V$.*

Proof. Every vector space of over Q which is a Z_2 -module is isomorphic to a direct sum $V^+ \oplus V^-$, of submodules such that $v \in V^+ \Leftrightarrow \sigma \cdot v = v$, and $v \in V^- \Leftrightarrow \sigma \cdot v = -v$. Since σ is in the center of ϕ , V^+ and V^- are ϕ -modules. But V is irreducible so $V = V^+$ or $V = V^-$. q.e.d.

We next construct the module A . $(C/2C)^\phi \approx \bigoplus (S_i/2S_i)^\phi$ where by assumption $C = \bigoplus S_i \cdot S_i$ is irreducible, so by the previous lemma each element of G either acts trivially or by negation on it. Thus, if we choose a complete set of representatives of $(C/2C)^\phi$ in C as generated by the union of complete sets of representatives of each $(S_i/2S_i)^\phi$, the resulting G -module A which they generate is a direct sum of one-dimensional G -submodules and projects surjectively onto $(C/2C)^\phi$.

Split $M \otimes_2 Q$ as $D \oplus (C \otimes Q)$. Lemma 3.7 permits us to reason exactly as in Lemma 3.3 to find a Z_2 -basis $\sigma_i, i = 1, m$, of the subgroup H of G of elements which act trivially on A and ϕ -irreducible direct summands $D_i, i = 1, m$, of D such that σ_i acts trivially on each $D_j \cap M$ excepting $D_i \cap M$ upon which it acts by negation. By our assumptions each $(D_i \cap M) \oplus C$ is not pure, so by the same arguments as in Lemma 3.3 there exists $u_j \in (D_i \oplus (C \otimes Q)) \cap M$ such that

$$\sigma_i \cdot u_j = \begin{cases} u_j, & j \neq i, \\ -u_j + w_j, & i = j, \end{cases}$$

where $w_j \in C - 2C$. Further, the vectors $\sigma_1, \dots, \sigma_m$ may be extended to a Z_2 -basis $\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_p$ where

$$\tau_i \cdot u_j \equiv u_j, \text{ mod } C.$$

Now $w_j \in C - 2C$ implies that the projection v_j of w_j to $C/2C$ is not zero. Since ϕ is a 2-group, the ϕ -module generated by v_j has nontrivial intersection with $(C/2C)^\phi - 0$. Since this module is just all ϕ -linear combinations of v_j there exists $f_j \in Z[\phi]$ such that $f_j(v_j)$ is in $(C/2C)^\phi - 0$. Set $y_j = f_j(u_j)$ and let

$L = \text{Span}_{\mathbb{Z}[G]}(A \cup \{y_j\text{'s}\})$. Since ϕ is central, σ_j commutes with f . Hence

$$(17) \quad \sigma_j \cdot y_j = f_j(\sigma_j \cdot u_j) = -f_j(u_j) + f_j(w_j) = -y_j + f_j(w_j).$$

Since $f_j(w_j)$ is a lift of $f_j(v_j)$ to C , it is a nonzero element of A , mod $2C$. The same type arguments give the following equations:

$$(18) \quad \sigma_i \cdot y_j = y_j, i \neq j; \tau_i \cdot y_j = y_j + z_{ij},$$

where z_{ij} is also an element of A mod $2C$. Since each element of G acts either trivially or by pure negation on each S_i , S_i splits as $(A \cap S_i) \oplus B_i$ where B_i is any \mathbb{Z} -complement of $A \cap S_i$. Clearly $L \cap B \subset 2B$. That A' is maximal in L' follows from (17). The semisimplicity of L'/A' follows from (17) and (18) together.

References

- [1] L. S. Charlap, *Compact flat Riemannian manifolds*, I, Ann. of Math. **81** (1965) 15–30.
- [2] L. S. Charlap & A. T. Vasquez, *Compact flat Riemannian manifolds*, II, Amer. J. Math. **87** (1965) 551–563.
- [3] P. E. Conner & E. E. Floyd, *Differentiable periodic maps*, Interscience, New York, 1962.
- [4] C. Curtis & I. Reiner, *Representation theory of finite groups and associative algebras*, Interscience, New York, 1962.
- [5] S. MacLane, *Homology*, Springer, New York, 1975.
- [6] J. Milnor & J. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, 1974.
- [7] C. Sah, *Abstract algebra*, Academic Press, New York, 1968.
- [8] A. T. Vasquez, *Flat Riemannian manifolds*, J. Differential Geometry **4** (1970) 367–381.
- [9] J. Wolf, *Spaces of constant curvature*, Publish or Perish, Boston, 1974.

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